Math 250 Briggs 2/e

7.4 Exponential Models

Objectives

1) Understand the difference between linear growth and exponential growth
   - calculate growth rate vs. relative growth rate
   - linear functions have constant growth rate
   - exponential functions have constant relative growth rate

2) Use exponential growth functions $y = y_0 e^{kt}$, $k > 0$
   - population
   - continuously compounded interest
   - others with constant relative growth rate

3) Use exponential decay functions $y = y_0 e^{kt}$, $k < 0$
   - radioactive decay
   - drug absorption in the body

Math 250

① Given the functions $f(t) = 2200 + 287.2t$
$g(t) = 1100(6.828)^{.05t}$

a) find the absolute growth rate
b) find the relative growth rate
c) find $\lim_{t\to\infty}\left(\frac{f'(t)}{f(t)}\right)$ and $\lim_{t\to 0}\left(\frac{g'(t)}{g(t)}\right)$

d) Complete table to be provided
e) graph $f(t)$ and $g(t)$ for $[0,40]$
f) graph $f'(t)$ and $g'(t)$ for $[0,40]$
g) graph $f'(t)/f(t)$ and $g'(t)/g(t)$ for $[0,40]$

a) Defn: Absolute growth rate = rate of change = derivative

$f'(t) = \frac{d}{dt}(2200 + 287.2t) = \boxed{287.2}$

$g'(t) = \frac{d}{dt}\left(1100(6.828)^{.05t}\right) = 1100 \cdot \ln(6.828) \cdot (6.828)^{.05t} \cdot (0.05)$

$= \boxed{55 \ln(6.828)(6.828)^{.05t}}$

b) Defn. Relative growth rate = growth rate as a percentage (decimal) of function value

$\boxed{\frac{f'(t)}{f(t)} = \frac{287.2}{2200+287.2t}}$

$\frac{g'(t)}{g(t)} = \frac{1100(6.828)^{.05t}(\ln 6.828)(0.05)}{1100(6.828)^{.05t}} = \boxed{0.05 \ln 6.828} \approx .096$

c) $\lim_{t\to\infty} \frac{287.2}{2200+287.2t} = \boxed{0}$

$\lim_{t\to\infty} 0.05\ln(6.828) = \boxed{0.05 * \ln(6.828)}$

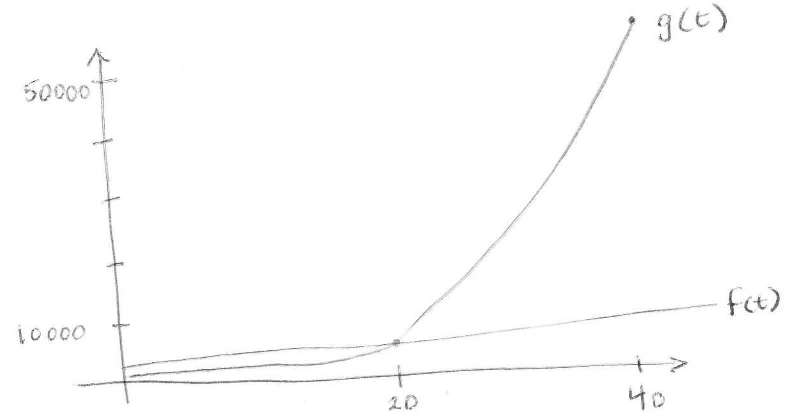
Some conclusions:
- $f(t)$ is a linear function. It has constant absolute growth rate, but its relative growth rate goes to 0 over time.
- $g(t)$ is an exponential function. Its absolute growth rate is also exponential, but its relative growth rate is constant.

d) Evaluate

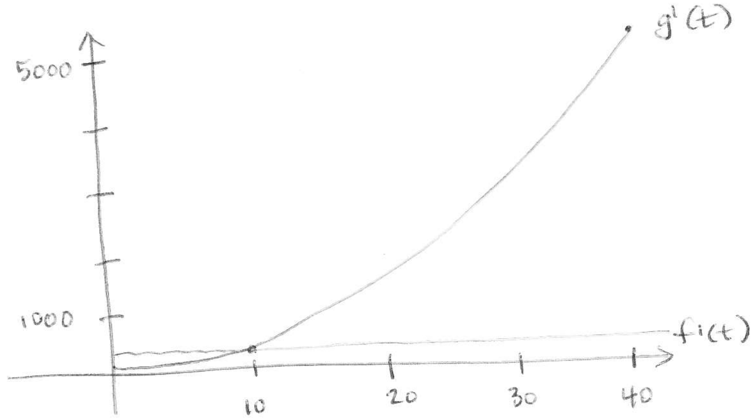
	$f(t)$ nearest whole	$g(t)$ nearest whole	$f'(t)$ nearest tenth	$g'(t)$ nearest tenth	$f'(t)/f(t)$ nearest hundredth	$g'(t)/g(t)$ nearest thousandth
t=0						
t=10						
t=20						
t=40						

d)	f(t)	g(t)	f'(t)	g'(t)	$\frac{f'(t)}{f(t)}$	$\frac{g'(t)}{g(t)}$
t=0	2200	1100	287.2	105.7	.13	.096
t=10	5072	2874	287.2	276.1	.06	.096
t=20	7944	7511	287.2	721.4	.04	.096
t=40	13688	51284	287.2	4925.9	.02	.096

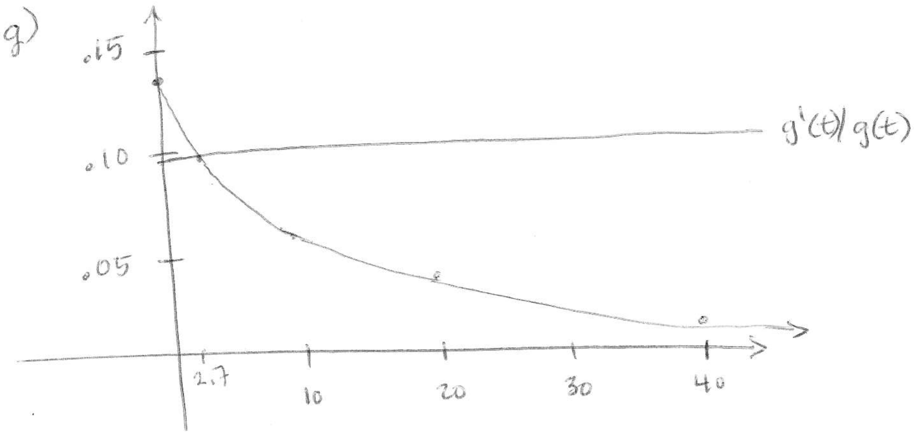
e)



f)



g)



Math 250

<u>Exponential Growth Curve</u> describes the growth of a population without considering limitations from the environment or predators. It is sometimes called "unrestricted growth."

$y(t) = y_0 e^{kt}$

y = population
t = time since initial population
k = relative growth rate (constant)
y_0 = initial population (constant)

(2) A population of rabbits is modeled by exponential growth, growing from 60 rabbits to 937 rabbits in 3 years.

a) Find the exact value of k.
b) Approximate k to the nearest hundredth.
c) (Extra credit preview) Write the exponential growth function with the exact k fully simplified to a base other than e.

a) $y = y_0 e^{kt}$

$937 = 60e^{k\cdot 3}$

$\frac{937}{60} = e^{3k}$

$\ln\left(\frac{937}{60}\right) = 3k$

$\boxed{k = \frac{1}{3}\ln\left(\frac{937}{60}\right)}$

$\boxed{k = \ln\sqrt[3]{\frac{937}{60}}}$

b) $k \approx .916$

$\boxed{k \approx .92}$

c) $y = 60e^{\left(\ln\sqrt[3]{\frac{937}{60}}\right)t}$

$y = 60\left(e^{\ln\sqrt[3]{\frac{937}{60}}}\right)^t$

$y = 60\cdot\left(\sqrt[3]{\frac{937}{60}}\right)^t$

or better

$\boxed{y = 60\cdot\left(\frac{937}{60}\right)^{\frac{1}{3}t}}$

<u>Exponent Laws</u>

$(x^n)^m = x^{n\cdot m}$

go backward

$x^{nm} = (x^n)^m$ or $(x^m)^n$

M250

(3) The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks does the tumor have 1500 cells?

Exponential growth $y = y_0 e^{kt}$

initial amount $y_0 = 8$

doubles to $y = 16$ when $t = 6$ weeks

$16 = 8e^{k \cdot 6}$ subst $y=16$ and $t=6$.

$\frac{16}{8} = e^{6k}$ isolate exponential

$2 = e^{6k}$

$\ln 2 = \ln e^{6k}$ take ln both sides

$\ln 2 = 6k$ inverse log property to simplify RHS

$\frac{\ln 2}{6} = k$ isolate k

$\frac{1}{6}\ln 2 = k$ power property of logs

$\ln \sqrt[6]{2} = k$

$y = 8e^{\ln \sqrt[6]{2}\, t} = 8(\sqrt[6]{2})^t = 8(2)^{t/6}$ subst into function

$y = 8(2)^{t/6}$

$y = 1500$, find t:

$1500 = 8(2)^{t/6}$ subst $y = 1500$

$\frac{1500}{8} = 2^{t/6}$ isolate exponential

$\frac{375}{2} = 2^{t/6}$ take logs both sides (any base allowed)

$\ln\left(\frac{375}{2}\right) = \ln\left(2^{t/6}\right)$ power property of logs

$\ln\left(\frac{375}{2}\right) = \frac{t}{6}\ln(2)$ isolate t

M250

$$\frac{6 \ln\left(\frac{375}{2}\right)}{\ln(2)} = t$$

$t \approx 45.3044$

$\boxed{t \approx 45.3 \text{ weeks}}$

Math 250

Exponential Decay

(4) After t years, the value of a car purchased for \$25,000 is $V(t) = 25000\left(\frac{3}{4}\right)^t$.

exponential $\left(\frac{3}{4}\right)^t$

base $\frac{3}{4} < 1$ means decay.

a) Determine the value of the car 2 years after it was purchased.

$$V(2) = 25000\left(\frac{3}{4}\right)^2$$

$$= 25000\left(\frac{9}{16}\right)$$

$$= \boxed{\$14062.50}$$

b) Find the rates of change of V with respect to t when t=1 and t=4. Give exact, then rounded to 2 places, with units.

rate of change $\Rightarrow$ derivative

w/ respect to t $\Rightarrow \frac{d}{dt}$.

$$V'(t) = 25000 \cdot \frac{d}{dt}\left(\frac{3}{4}\right)^t$$

$$V'(t) = 25000 \cdot \ln\left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right)^t$$

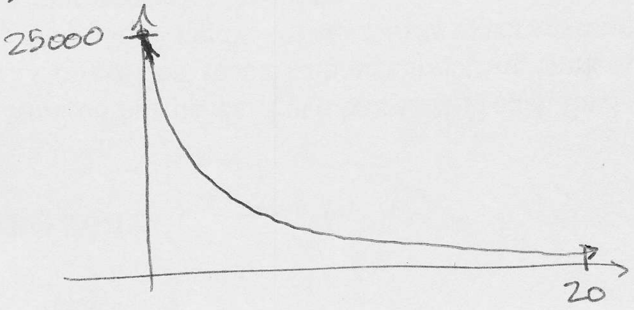
$$V'(1) = 25000 \cdot \ln\left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right)^1 = 18750 \ln\left(\frac{3}{4}\right)$$

$$\approx \boxed{-\$5394.04/\text{yr}}$$

$$V'(4) = 25000 \cdot \ln\left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right)^4 = \frac{253125}{32} \cdot \ln\left(\frac{3}{4}\right)$$

$$\approx \boxed{-\$2275.61}$$

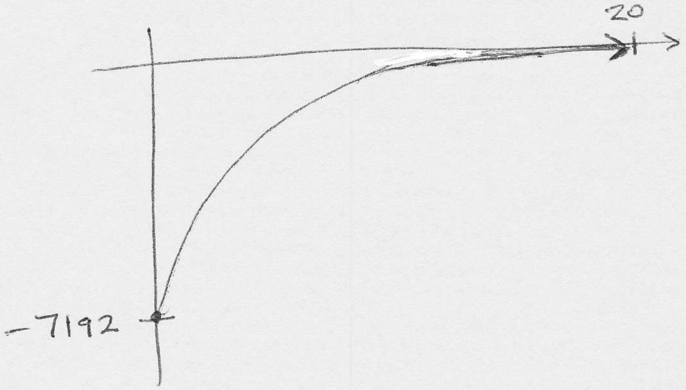
c) Look at graph of V(t) in GC. What can be said about V'(t) by looking at V(t).



- It's always negative.
- At first it's a very large negative #, but gradually increases.

Math 250

d) Look at graph of $V'(t)$ in GC.
Determine the horizontal asymptote and explain what this means.



horizontal asymptote $y = 0$.

rate of change of value never quite reaches 0, but it approaches it.

The car loses value <u>very</u> rapidly when it is new, but very slowly when it is old.

Compound Interest

Interest compounded n times per year

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = amount in account after t years

P = initial amount in account
Principal

r = interest rate per year

n = # times compounded per year

"Compounding" means calculating interest and adding it to the principal so that it earns interest.

Compound interest formula is used
1) when money is loaned/borrowed
2) when money is invested/saved.

Interest is the cost of borrowing money.
An investor is allowing the bank to borrow money,
So the bank pays interest.
A borrower gets a loan and repays the loan plus the interest (additional cost for borrowing).

If we take limit:

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = \boxed{Pe^{rt} = A}$$

continuous compounding

To calculate this limit, we need

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

which is proven in 7.2 (p437)

5) Find A when $P = \$2500$, $r = 6\%$, $t = 20$ years, a) $n = 12$ b) continuously

a) $A = P\left(1+\frac{r}{n}\right)^{nt} = 2500\left(1+\frac{.06}{12}\right)^{12(20)} = \boxed{\$8275.51}$

b) $A = Pe^{rt} = 2500\,e^{(.06\times 20)} = \boxed{\$8300.29}$

M250

*** Caution ***

when using GC to calculate, must use ( ) around exponent

2500(1+0.06/12)^(12*20)

order of operations must be changed to multiply before exponent.

⑥ How much money should be invested at 3% compounded a) daily (n=365) for 30 years to result in $500,000? b) continuously

$A = 500{,}000$

$P = ?$

$r = 0.03$

$n = 365$

$t = 30$

a) $500000 = P\left(1+\frac{0.03}{365}\right)^{(365)(3)}$

$$\frac{500000}{\left(1+\frac{0.03}{365}\right)^{(365\times30)}} = P$$

$$\boxed{P = \$203{,}292.35}$$

b) $500000 = Pe^{(.03)(30)}$

$$\frac{500000}{e^{.03*30}} = P$$

$$P = \boxed{\$203{,}284.83}$$